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# Localization of Electromagnetic and Acoustic Waves in Random Media. Lattice Models

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We consider lattice versions of Maxwell's equations and of the equation that governs the propagation of acoustic waves in a random medium. The vector nature of electromagnetic waves is fully taken into account. The medium is assumed to be a small perturbation of a periodic one. We prove rigorously that localized eigenstates arise in a vicinity of the edges of the gaps in the spectrum. A key ingredient is a new Wegner-type estimate for a class of lattice operators with off-diagonal disorder.

**KEY WORDS:** Localization; random media; electromagnetic waves; acoustic waves; lattice model.

## 1. INTRODUCTION

Decades after Anderson[1] described the remarkable phenomenon of the localization in space of electron wave functions in disordered solids, physicists have begun to ask whether other waves, say electromagnetic or acoustic, can be localized if the propagating if the propagating media is disordered appropriately.[2,3] It is well known that the rise of localized eigenstates in disordered media and the rise of gaps in the spectrum for periodic media are intimately related phenomena, both due to multiple scattering and interference of waves,[4] and should be studied simultaneously. Thus, if a periodic medium exhibits gaps in the spectrum and then it is slightly disordered, one can expect the rise of localized eigenmodes with energies in a vicinity of the edges of the gaps. Physical arguments and numerical computations, as well as experiments, indicate

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the possibility of a gap regime for periodic two-component media. The most recent theoretical and experimental achievements in the investigation of photonic band-gap structures are published in a series of papers in ref. 5. Nevertheless, some theoretical arguments and experimental evidence suggest that the existence of gaps and localization for dielectrics and acoustic media are not easy to achieve,[6-9] i.e., the parameters of such media ought to be carefully calculated. In particular, high contrast in two-component media and some shapes of atoms of the embedded material favor band-gap regime and localization.

The objective of this article is to give a rigorous proof of the existence of exponentially localized eigenstates for lattice models of disordered dielectric and acoustic media. The disordered media we consider here are assumed to be small random perturbations of periodic ones. The relation of the models we introduce to the "true" continuous models is similar to the relation of the Anderson tight-binding lattice model to the Schrödinger operator. We shall assume here that the initial periodic medium possesses a gap in the spectrum, since the existence of gaps in the spectrum for two-component media is proved in ref. 10 if the contrast in the dielectric constant (or the corresponding coefficient for the acoustic waves) between two components is large enough. Using some techniques from refs. 11-13, we prove then the existence of exponentially localized states in a vicinity of the edges of the gaps in the spectrum. A key ingredient is a new Wegner-type estimate for a class of lattice operators with off-diagonal disorder.

Basic properties of wave propagation in a nonhomogeneous medium eventually boil down to the spectral properties of the relevant self-adjoint differential operators with coefficients varying in the space. These operators for electromagnetic and elastic waves have, respectively, the forms

$$\Lambda\Psi = \nabla \times (\gamma(x)\nabla \times \Psi), \qquad \gamma(x) = \varepsilon^{-1}(x), \qquad x \in \mathbf{R}^3 \tag{1}$$

$$\Gamma\psi = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\gamma(x)\frac{\partial}{\partial x_j}\psi, \qquad x \in \mathbf{R}^d \tag{2}$$

According to the philosophy of Anderson localization,[1] if the coefficient $\gamma(x)$ is a random field and the operator $\Lambda$ or $\Gamma$ has gaps in the spectrum, localized states can appear in a vicinity of the edges of the gaps in the spectrum. We justify this philosophy for the lattice versions of operators $\Lambda$ and $\Gamma$. Namely, we consider here the discrete analogs of these operators by replacing the operations of differentiation by their finite-difference counterparts. From now on the symbols $\Lambda$ and $\Gamma$ will refer to the lattice versions of the corresponding operators in (1) and (2). We shall assume:

(i) The random coefficient $\gamma(x)$ is a small perturbation of a periodic one, $\gamma_0(x)$.

(ii) $\gamma_0(x)$ is such that the operator $\Lambda_0$ or $\Gamma_0$ has gaps in the spectrum.

Under these conditions we prove the existence of localized states, i.e., pure point spectrum, with probability 1 for the operators $\Lambda$ or $\Gamma$, in a vicinity of the edges of the gaps of these operators. The assumption (ii) above is fulfilled in physically interesting cases.[10]

The disorder associated with the lattice operators $\Lambda$ and $\Gamma$ is a type of off-diagonal disorder. A more restrictive type of off-diagonal disorder was studied in refs. 14 and 15, where exponential localization at high energies is proven. Their random operators are sums of independent random rank-one operators, while the operators in the class studied in this paper (including $\Lambda$ and $\Gamma$) are sums of independent random operators of a fixed (but arbitrary) finite rank.

**Remark 1.** There is much similarity between the spectral properties (and their proofs) of the operators $\Lambda$ and $\Gamma$ and of the lattice Schrödinger operator we considered in ref. 11. But since there has been doubt about whether this is true, especially in the case of the Maxwell operator $\Lambda$, which acts on *vector-valued* wave functions, we set down all necessary estimates for this case, including those that are almost the same as for the Schrödinger operator, in order to be perfectly sure that nothing is missed.

## 2. STATEMENT OF RESULTS

We begin with the construction of the lattice operators $\Lambda$ and $\Gamma$. In order to do this, we first introduce discrete analogs of the partial derivatives $\partial_j$ and $\nabla$ as follows. Let $V_j$, $1 \leqslant j \leqslant d$ ($d$ is the dimension of the space, i.e., 3 in many interesting cases), be the unitary shift operators acting on the Hilbert spaces $l^2(\mathbf{Z}^d)$ of $l^2(\mathbf{Z}^3, \mathbf{C}^3)$, which are respectively the spaces of $\mathbf{C}$- or $\mathbf{C}^3$-valued functions $\psi$ on the lattice $\mathbf{Z}^d$ or $\mathbf{Z}^3$ with the scalar product given by $(\varphi, \psi) = \sum_x \varphi^*(x)\, \psi(x)$. If $e_j$, $1 \leqslant j \leqslant d$, are the standard basis vectors in lattice $\mathbf{Z}^d$ and $I$ is the identity operator, then $V_j$ and $\partial_j$ are defined by

$$\partial_j = I - V_j, \quad (V_j \Psi)(x) = \Psi(x - e_j), \qquad x \in \mathbf{Z}^d, \quad 1 \leqslant j \leqslant d \tag{3}$$

We define the lattice version of $\nabla$ by substituting the partial derivatives by their lattice counterparts $\partial_j$ defined in (3). That is, the lattice analogs of operators defined by (1) and (2) have respectively the forms

$$\Lambda\Psi = \nabla^* \times (\gamma(x)\nabla \times \Psi), \qquad \gamma(x) = \varepsilon^{-1}(x), \qquad x \in \mathbf{Z}^3 \tag{4}$$

$$\Gamma\psi = -\sum_{j=1}^{d} \partial_j^* \gamma(x)\, \partial_j \psi, \qquad x \in \mathbf{Z}^d \tag{5}$$

where the action of the operator $\nabla^*$ is defined in terms of the corresponding action of the operators $\partial_j^*$, $1 \leqslant j \leqslant d$ (see ref. 10 for more details).

Let us consider first the case of a periodic medium, i.e., the case when $\gamma(x) = \gamma_0(x)$ is a periodic function of $x, x \in \mathbf{Z}^d$, with the corresponding operators $\Lambda_0$ and $\Gamma_0$ being defined by (4), (5), and (3) (we will refer to these operators loosely as periodic operators). Thus, we suppose that these exists $q = (q_1,..., q_d) \in \mathbf{Z}^d$ such that

$$\gamma_0(x + \alpha q) = \gamma_0(x), \qquad \forall x, \alpha = (\alpha_1,..., \alpha_d) \in \mathbf{Z}^d, \qquad \alpha q = (\alpha_1 q_1,..., \alpha_d q_d) \tag{6}$$

and will call any function $\gamma_0$ satisfying (6) $q$-periodic. Clearly a $q$-periodic function $\gamma_0(x)$ is uniquely defined by its values on the parallelepiped

$$\mathscr{Q} = \{0,..., q_1 - 1\} \times \cdots \times \{0,..., q_d - 1\} \subset \mathbf{Z}^d \tag{7}$$

The lattice operators $\Lambda_0$ and $\Gamma_0$ are particular examples of the lattice local periodic operators we studied in ref. 11. Thus, the following statement holds for the operators $\Lambda_0$ and $\Gamma_0$.

**Proposition 1** (Band structure of spectrum). If $\gamma_0(x)$ is a periodic function, then the spectrum $\sigma_0$ of the operator $\Gamma_0$ (or $\Lambda_0$) consists of a finite number $J$ of intervals, namely

$$\sigma_0 = \bigcup_{1 \leqslant j \leqslant J} [\mu_j^{(0)}, \lambda_j^{(0)}]; \qquad 0 \leqslant \mu_j^{(0)} \leqslant \lambda_j^{(0)}, \qquad 1 \leqslant j \leqslant J$$
$$\lambda_j^{(0)} < \mu_{j+1}^{(0)}, \qquad 1 \leqslant j \leqslant J-1 \tag{8}$$

**Remark 2.** We call the intervals above bands. If $J > 1$, then clearly we have gaps in the spectrum: $[\lambda_j^{(0)}, \mu_{j+1}^{(0)}]$, $1 \leqslant j \leqslant J-1$.

An example of a periodic medium exhibiting gaps is constructed in ref. 10. Since we are interested here primarily in the case of random $\gamma(x)$, we will just assume the existence of gaps for the initial unperturbed periodic medium.

**Assumption 1.** $\gamma_0(x)$ is a real-valued $q$-periodic function of $x, x \in \mathbf{Z}^d$, such that $0 < c_0 \leqslant \gamma_0(x) \leqslant c_1 < \infty$ and the corresponding operator $\Gamma_0$ (or $\Lambda_0$) has at least one gap in the spectrum.

We introduce the random coefficient $\gamma(x)$, which is a perturbation of the periodic $\gamma_0(x)$, as follows:

$$\gamma(x)=\gamma_0(x)[1+g\xi(x)] \tag{9}$$

where the positive constant $g$ and the random field $\xi(x)$ satisfy in turn the following assumption.

**Assumption 2.** $\xi(x)$, $x\in\mathbf{Z}^d$, are independent, identically distributed random real-valued variables on a probability space with probability measure $\mathbf{P}$. The probability distribution of $\xi(0)$ has a density $\rho$ with $\|\rho\|_\infty\leqslant D_0<\infty$. There exist constants $\xi_1$ and $\xi_2$ such that

$$-\infty<\xi_1<0<\xi_2<\infty \qquad \text{and} \qquad \mathscr{R}_{\xi(x)}=[\xi_1,\xi_2] \tag{10}$$

where by $\mathscr{R}_\xi$ we denote the essential range of the random real-valued variable $\xi$.

In addition, in order to keep $\gamma(x)$ positive, the constant $g$ is small enough, namely

$$1+g\xi_1>0 \tag{11}$$

**Theorem 1** (Location of the spectrum). Suppose that $\gamma(x)$ is defined by (9), where $\gamma_0(x)$ and $\xi(x)$, $g$ satisfy Assumptions 1 and 2, respectively. Then the following statements hold:

(i) With probability 1 the spectrum $\sigma(\Lambda)$ of the operator $\Lambda$ [or $\sigma(\Gamma)$] is nonrandom, i.e., there exists a closed set $\sigma\subseteq\mathbf{R}$ such that with probability 1, $\sigma(\Lambda)=\sigma$ [or $\sigma(\Gamma)=\sigma$]; in addition, if $\sigma_0$ is the spectrum of the operator $\Lambda_0$ (or $\Gamma_0$), then the following representation is true:

$$\sigma=\bigcup_{\xi_1\leqslant t\leqslant\xi_2}(1+gt)\,\sigma_0 \tag{12}$$

(ii) If we use the notations of Proposition 1 and introduce $g_j$ by the equality

$$\lambda_j^{(0)}(1+g_j\xi_2)=\mu_{j+1}^{(0)}(1+g_j\xi_1), \qquad 1\leqslant j\leqslant J-1 \tag{13}$$

then for any $0\leqslant g<g_j$ the spectrum $\sigma$ has a nonempty gap

$$]\lambda_j,\mu_{j+1}[, \qquad \lambda_j=\lambda_j^{(0)}(1+g\xi_2)<\mu_{j+1}=\mu_{j+1}^{(0)}(1+g\xi_1) \tag{14}$$

where $\lambda_j$, $\mu_{j+1}\in\sigma$. This gap is associated naturally with the gap $]\lambda_j^{(0)},\mu_{j+1}^{(0)}[$ in the spectrum of the unperturbed periodic operator.

In other words, Theorem 1 tells us that the spectrum of the random operator is nonrandom and has a band structure. Moreover, taking the coefficient $g$ small enough, we can open up any gap in the spectrum which is associated with the unperturbed periodic operator.

Another object which we shall need to study in order to establish exponential localization is the integrated density of states $N(d\lambda)$ associated with the random operators $\Lambda$ and $\Gamma$. More precisely, we will need a Wegner-type estimate for its density $N(d\lambda)/d\lambda$.

Some notation: We write

$$V^x = \prod_{1 \leqslant j \leqslant d} V_j^{x_j}, \qquad x = (x_1,..., x_d), \qquad x \in \mathbf{Z}^d \tag{15}$$

We will denote by $e_{\alpha,x}$, $x \in \mathbf{Z}^d$, $\alpha = 1, 2,..., D$, the standard basis in the space $l^2(\mathbf{Z}^d, \mathbf{C}^D)$, i.e., $e_{\alpha,x}(\alpha, x) = 1$ and $e_{\alpha,x}(\beta, y) = 0$ if $\beta \neq \alpha$ or $y \neq x$. If $D = 1$, we omit $\alpha$. Given an operator $A$ in the Hilbert space $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ and $\mathcal{O} \subseteq \mathbf{Z}^d$, we denote by $A_{\mathcal{O}}$ the operator on $l^2(\mathcal{O}, \mathbf{C}^D)$ given by the restriction of $A$ to $\mathcal{O}$ with Dirichlet boundary conditions, i.e., with matrix elements $A_{\mathcal{O}}((\alpha, x), (\beta, y)) = A((\alpha, x), (\beta, y))$ for all $x, y \in \mathcal{O}$ and $\alpha, \beta = 1, 2,..., D$. We shall write $|\mathcal{O}|$ for the number of elements in $\mathcal{O}$ and $|x|_\infty = \max_{1 \leqslant j \leqslant d} |x_j|$. In addition, we write

$$\mathcal{O}_s = \{x \in \mathbf{Z}^d; \operatorname{dist}(x, \mathcal{O}) \leqslant 2s\} \tag{16}$$

Notice that $|\mathcal{O}_s| \leqslant (4s+1)^d \, |\mathcal{O}|$.

We have the following general result:

**Theorem 2** (Wegner-type estimate). Let $B_0$ be a nonnegative operator of finite rank $r$ in the Hilberty space $l^2(\mathbf{Z}^d, \mathbf{C}^D)$, with $(B_0 e_{\alpha,x}, e_{\beta,y}) = 0$ unless we have $|x|_\infty$, $|y|_\infty \leqslant s$ for a given $s < \infty$, and let $B_x = V^x B_0 V^{-x}$, $x \in \mathbf{Z}^d$. Let $A$ be the operator on $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ defined by

$$A = \sum_{x \in \mathbf{Z}^d} t_x B_x \tag{17}$$

on functions with finite support in $\mathbf{Z}^d$, where the $t_x$, $x \in \mathbf{Z}^d$, are nonnegative random variables forming a real-valued metrically transitive field on $\mathbf{Z}^d$ (see ref. 16; for instance, they can be independent and identically distributed), such that

$$\mathbf{E}\{t_0^2\} > \infty \tag{18}$$

and the conditional probabilities $p_x(dt) = p_x(dt, \cdot) = \mathbf{P}\{t_x \in dt \mid t_y, \ y \neq x\}$ satisfy

$$p_x(dt) \leqslant K(t)\, dt \qquad \text{for any} \quad x \in \mathbf{Z}^d \tag{19}$$

for some nonnegative measurable function $K(t)$ on $(0, \infty)$, with

$$K = \sup_{0 < t < \infty} tK(t) < \infty \tag{20}$$

Then $A$ is a nonnegative, essentially self-adjoint operator with probability 1 and, if $E(A, d\lambda)$ is its resolution of identity, i.e., $A = \int_{\lambda \in \mathbf{R}} \lambda E(A, d\lambda)$, and

$$N(A, d\lambda) = \mathbf{E}\left\{D^{-1} \sum_{1 \leqslant \alpha \leqslant D} (E(A, d\lambda)\, e_{\alpha,0}, e_{\alpha,0})\right\} \tag{21}$$

its integrated density of states, then the density of states is estimated as follows:

$$\frac{N(A, d\lambda)}{d\lambda} \leqslant \frac{rCT}{\lambda}, \qquad \lambda > 0 \tag{22}$$

Moreover, for all finite $\mathcal{O} \subseteq \mathbf{Z}^d$ and $0 < \varepsilon < \lambda$ we have

$$\mathbf{P}\{\operatorname{dist}(\lambda, \sigma(A_{\mathcal{O}})) \leqslant \varepsilon\} \leqslant \frac{2\varepsilon rCT\, |\mathcal{O}_s|}{\lambda - \varepsilon} \tag{23}$$

In particular, if Assumption 1 is fulfilled, then the operators $A$ and $\Gamma$ are of the form (17) with $r = 3$, $s = 1$ and $r = d$, $s = 1$, respectively, so (22) and (23) hold for these operators.

The main result of this paper is the following:

**Theorem 3** (Localization). Suppose the hypothesis of Theorem 1 are satisfied and let us keep the notations of the theorem. Assume that for a given $j$, $1 \leqslant j \leqslant J-1$, the constant $g$ satisfies the inequality $0 \leqslant g < g_j$. Let us pick numbers $v_\pm$ such that $\xi_1 < v_- < 0 < v_+ < \xi_2$. Then there exists $\tilde{p}_+ > 0$ (or respectively $\tilde{p}_- > 0$) such that if

$$p_+ = \mathbf{P}\{\xi(x) \in [v_+, \xi_2]\} < \tilde{p}_+ \qquad (p_- = \mathbf{P}\{\xi(x) \in [\xi_1, v_-]\} < \tilde{p}_-) \tag{24}$$

we can find $\delta_+ > 0$ ($\delta_- > 0$) such that the interval $I_j^+$ ($I_j^-$) defined by

$$I_j^+ = ]\lambda_j - \delta_+, \lambda_j[ \qquad (I_j^- = ]\mu_{j+1}, \mu_{j+1} + \delta_-[) \tag{25}$$

belongs to the nonrandom spectrum $\sigma$ and with probability 1 the spectrum of $\Gamma$ (or $A$) on this interval is purely pure point and the corresponding eigenfunctions decay exponentially at infinity. Moreover,

$$\lim_{p_+ \to 0} \delta_+ = g(\xi_2 - v_+)\, \lambda_j^{(0)} \qquad (\lim_{p_- \to 0} \delta_- = g(v_- - \xi_2)\, \mu_{j+1}^{(0)}) \tag{26}$$

We also prove a somewhat different version of Theorem 3 based on different assumptions imposed on the random field $\xi$.

**Theorem 4** (Localization). Suppose the hypotheses of Theorem 1 are satisfied and let us keep the notations of that theorem. Assume that for a given $j$, $1 \leqslant j \leqslant J-1$, the constant $g$ satisfies the inequality $0 \leqslant g < g_j$. Suppose that for all $\varepsilon > 0$

$$\mathbf{P}\{\xi_2 - \xi(x) \leqslant \varepsilon\} \leqslant C\varepsilon^\eta \qquad (\mathbf{P}\{\xi(x) - \xi_1 \leqslant \varepsilon\} \leqslant C\varepsilon^\eta) \tag{27}$$

for some $C < \infty$ and $\eta > d$. Then there exists $\delta_+ > 0$ ($\delta_- > 0$) such that the interval $I_j^+$ ($I_j^-$) defined by

$$I_j^+ = ]\lambda_j - \delta_+, \lambda_j[ \qquad (I_j^- = ]\mu_{j+1}, \mu_{j+1} + \delta_-[) \tag{28}$$

belongs to the nonrandom spectrum $\sigma$ and with probability 1 the spectrum of $\Gamma$ (or $\Lambda$) on this interval is purely pure point and the corresponding eigenfunctions decay exponentially at infinity.

The proofs of Theorems 3 and 4 are similar to the analogous results for the Schrödinger operators with diagonal disorder in ref. 11; in particular, it employs the multiscale analysis methods from refs. 12 and 13.

## 3. PROOF OF THEOREMS 1 AND 2, AND AUXILIARY STATEMENTS

In this section we investigate the location of the spectrum of the operators $\Lambda$ and $\Gamma$. Many statements we shall consider are formulated and proved in a uniform way for both operators $\Lambda$ and $\Gamma$. For this reason we will use the symbol $\mathfrak{A}$ to denote either of them. In addition, whenever we shall need to emphasize that $\mathfrak{A}$ depends on $\gamma$ we write $\mathfrak{A}(\gamma)$. In order to simplify the notations we introduce also the periodic operator $\mathfrak{a} = \mathfrak{A}(\gamma_0)$ which corresponds to either $\Lambda_0$ or $\Gamma_0$.

### 3.1. Location of the Spectrum

Some statements we prove here are based on the results established in ref. 11. So we describe first some concepts introduced in ref. 11, applying them to the operators $\mathfrak{a}$ and $\mathfrak{A}$, which act in Hilbert space $l^2(\mathbf{Z}^d, \mathbf{C}^D)$, where for the operator $\Lambda$ we have $d = 3$ and $D = 3$, and for the operator $\Gamma$ we have $D = 1$. The operator $\mathfrak{A}$ is local and bounded.[11]

**Definition 1.** Let $u, v \in \mathbf{N}^d$. If $v = nu$ for some $n \in \mathbf{N}^d$, we will write $u \preccurlyeq v$. If in addition all the coordinates of $n$ are strictly greater than 1, we will write $u \prec v$.

**Defintion 2.** For $u \in \mathbf{N}^d$ we define a parallelepiped $C^u = \{0,..., u_1 - 1\} \times \cdots \times \{0,..., u_d - 1\} \subseteq \mathbf{Z}^d$. We will write $C^u \preccurlyeq C^v$ or $C^u \prec C^u$ if $u \preccurlyeq v$ or $u \prec v$, respectively.

Following ref. 11, we introduce for the operator $\mathfrak{a}$ and any $C^u \succ C^q$ the finite matrix

$$\mathring{\mathfrak{a}}_{C_u}((\alpha, x), (\beta, y)) = \sum_{n \in \mathbf{Z}^d} \mathring{\mathfrak{a}}_{C_u}((\alpha, x), (\beta, y + nu)) \tag{29}$$

$$x, y \in C^u, \qquad \alpha, \beta = 1,..., D$$

Applying Theorem 4 from ref. 11 to the operators $\mathfrak{A}(\xi)$, we obtained the following statement.

**Lemma 1.** Let $\theta(x)$, $x \in \mathbf{Z}^d$, be a $u$-periodic, positive, real-valued function. Suppose that $C_n$, $n = 1, 2,...$, is a sequence of parallelepipeds such that $C^u \preccurlyeq C_n \prec C_{n+1}$, $n \geqslant 1$. Then

$$\sigma[\mathring{\mathfrak{A}}_{C_n}(\theta)] \subseteq \sigma[\mathring{\mathfrak{A}}_{C_{n+1}}(\theta)] \tag{30}$$

$$\sigma[\mathfrak{A}(\theta)] = \overline{\bigcup_{n \geqslant 1} \mathring{\mathfrak{A}}_{C_n}(\theta)}, \qquad \sigma[\mathring{\mathfrak{A}}_{C_n}(\theta)] \subseteq \sigma[\mathfrak{A}(\theta)] \tag{31}$$

We shall also need the following form of Weyl's criterion.[17]

**Lemma 2** (Distance to the spectrum). Let $\mathscr{H}$ be a separable Hilbert space (in particular finite-dimensional space) and $A$ be a self-adjoint operator in $\mathscr{H}$. Then if $\sigma(A)$ is the spectrum of $A$ and $\lambda$ is a real number, then

$$\operatorname{dist}\{\sigma(A), \lambda\} = \min_{\psi \in \mathscr{H}, \|\psi\| = 1} = \|(A - \lambda)\psi\| \tag{32}$$

**Lemma 3.** Suppose that $C_n$, $n = 1, 2,...$, is a sequence of parallelepipeds such that $C^q \preccurlyeq C_n \prec C_{n+1}$, $n \geqslant 1$. Let $\mathscr{P}_q$ be the set of real-valued functions $\eta(x)$ on the lattice, each such function being $u$-periodic for some $u \succ q$ and satisfying $\xi_1 \leqslant \eta(x) \leqslant \xi_2$. Then there exists a nonrandom set $\sigma \subseteq \mathbf{R}$ such that with probability 1, $\sigma[\mathfrak{A}(\gamma)] = \sigma$ and the following representation is true:

$$\sigma = \overline{\bigcup_{\eta \in \mathscr{P}_q} \sigma[\mathfrak{A}((1 + g\eta)\gamma_0)]} = \overline{\bigcup_{n \geqslant 1, \eta \in \mathscr{P}_q} \sigma[\mathring{\mathfrak{A}}_{C_n}((1 + g\eta)\gamma_0)]} \tag{33}$$

*Proof.* We notice that the random operator $\mathfrak{A}$, i.e., either of the operators $A$ and $\Gamma$, is metrically transitive and therefore there exists a nonrandom set $\sigma \subseteq \mathbf{R}$ such that with probability 1, $\sigma[\mathfrak{A}(\gamma)] = \sigma$.[16] In other words, if $\Omega$ is the underlying probability space, there exists $\Omega_1 \subseteq \Omega$ with $\mathbf{P}(\Omega_1) = 1$ such that

$$\sigma[\mathrm{A}(\gamma_\omega)] = \sigma \qquad \text{for all} \quad \omega \in \Omega_1 \tag{34}$$

Then from Lemma 1 we obviously have

$$\overline{\bigcup_{\eta \in \mathscr{P}_q} \sigma[\mathfrak{A}((1+g\eta)\gamma_0)]} = \overline{\bigcup_{n \geqslant 1, \eta \in \mathscr{P}_q} \sigma[\mathring{\mathfrak{A}}_{C_n}((1+g\eta)\gamma_0)]} \tag{35}$$

Let us pick any positive $\varepsilon$ and an $\omega$ for which (34) is true. Assume that $\lambda \in \sigma$. Then in view of Lemma 2 there exist a natural $m$ and a vector $\psi$ in the Hilbert space such that $\|\psi\| = 1$ and

$$\|(\mathfrak{A}(\gamma_\omega) - \lambda)\,\psi(x)\| \leqslant \varepsilon, \qquad \psi(x) = 0, \qquad x \notin C_m \tag{36}$$

We may impose in (36) the extra constraints $\psi(x) = 0$, $x \notin C_m$, on the vector $\psi$ since the operator $\mathfrak{A}$ is local and bounded. Then for any $n > m$

$$\mathfrak{A}(\gamma_\omega)\,\psi(x) = \mathring{\mathfrak{A}}_{C_n}(\gamma_\omega)\,\psi(x), \qquad x \in C_n \tag{37}$$

and, therefore, treating $\psi$ as finite-dimensional vector in the range of the action of the finite matrix $\mathring{\mathfrak{A}}_{C_n}(\gamma_\omega)$, we obtain

$$\|(\mathring{\mathfrak{A}}_{C_n}(\gamma_\omega) - \lambda)\,\psi(x)\| \leqslant \varepsilon \tag{38}$$

Since $\varepsilon$ is an arbitrary positive number, the last inequality clearly implies that

$$\lambda \in \overline{\bigcup_{n \geqslant 1, \eta \in \mathscr{P}_q} \sigma[\mathring{\mathfrak{A}}_{C_n}((1+g\eta)\gamma_0)]} \tag{39}$$

and consequently

$$\sigma \subseteq \overline{\bigcup_{n \geqslant 1, \eta \in \mathscr{P}_q} \sigma[\mathring{\mathfrak{A}}_{C_n}((1+g\eta))\gamma_0)]} \tag{40}$$

To prove the opposite inclusion, let us pick again a positive $\varepsilon$ and a $u$-periodic $\eta \in \mathscr{P}_q$. Then we suppose that $\lambda \in \sigma[\mathfrak{A}\{(1+g\eta)\gamma_0\}]$. Since the operator $\mathfrak{A}$ is local and bounded, we can apply again Lemma 2 and state there exists a vector $\psi$, $\|\psi\| = 1$, such that

$$\|(\mathfrak{A}((1+g\eta)\gamma_0) - \lambda)\,\psi(x)\| \leqslant \varepsilon, \qquad \psi(x) = 0, \qquad x \notin C_m \tag{41}$$

Now we notice that in view of Assumption 2 for any positive $\delta$ there exists a set $\Omega_\eta$, $\mathbf{P}(\Omega_\eta)=1$, such that

$$\forall\delta, \forall\omega\in\Omega_\eta: \quad \exists a=a(\delta,\omega)\succcurlyeq u: \quad \max_{x\in C_m+a} |\xi_\omega(x)-\eta(x)|\leqslant\delta \tag{42}$$

Besides, if we denote $\psi_a(x)=\psi(x-a)$, then since $\eta$ is $u$-periodic, we have from (41)

$$\forall a\succcurlyeq u: \quad \|(\mathfrak{A}((1+g\eta)\gamma_0)-\lambda)\,\psi_a(x)\|\leqslant\varepsilon \tag{43}$$

Clearly, if we pick $\delta$ small enough, then $\forall\omega\in\Omega_\eta$: $\exists a\succcurlyeq u$:

$$\forall\omega\in\Omega_\eta: \quad \exists a=a(\varepsilon,\omega)\succcurlyeq u: \quad \|(\mathfrak{A}(\gamma_\omega)-\lambda)\,\psi_a(x)\|\leqslant 2\varepsilon \tag{44}$$

From this we immediately obtain

$$\sigma\supseteq\sigma[\mathfrak{A}((1+g\eta)\gamma_0)], \qquad \eta\in\mathscr{P}_q \tag{45}$$

and consequently

$$\sigma\supseteq\overline{\bigcup_{\eta\in\mathscr{P}_q}\sigma[\mathfrak{A}((1+g\eta)\gamma_0)]} \tag{46}$$

Thus, (35), (40), and (46) imply the desired relations (33), which completes the proof of the lemma. ∎

Let us introduce for a pair of real numbers $\eta_1\leqslant\eta_2$ the following notation:

$$\sigma(\eta_1,\eta_2)=\bigcup_{\eta_1\leqslant t\leqslant\eta_2}(1+gt)\,\sigma_0, \qquad \sigma_0=\sigma[\mathfrak{A}(\gamma_0)] \tag{47}$$

Now we can prove the following representation for the spectrum of a periodic operator.

**Lemma 4.** Suppose that $\eta$ is a $u$-periodic function, $u\succ q$, and $\xi_1\leqslant\eta_1\leqslant\eta(x)\leqslant\eta_2\leqslant\xi_2$. Then if $C\succ C^u$, the following is true:

$$\sigma[\mathring{\mathfrak{A}}_C((1+g\eta)\gamma_0)]\subseteq\bigcup_{\eta_1\leqslant t\leqslant\eta_2}(1+tg)\,\sigma[\mathring{\mathfrak{A}}_C(\gamma_0)]\subseteq\sigma(\eta_1,\eta_2) \tag{48}$$

In addition, with probability 1

$$\sigma[\mathfrak{A}(\gamma)]\subseteq\sigma(\xi_1,\xi_2) \tag{49}$$

*Proof.* The proof of the lemma is based on the min-max principle[18] formulated below. ∎

**Proposition 2** (Min-max principle). Let $A \leqslant B$ be two self-adjoint $N \times N$ matrices and let $\mu_n(A)$, $\mu_n(B)$, $1 \leqslant n \leqslant N$, be the respective eigenvalues listed in nondecreasing order. Then

$$\mu_n(A) \leqslant \mu_n(B), \qquad 1 \leqslant n \leqslant N \tag{50}$$

Let us notice now that

$$\mathring{\mathfrak{A}}_C((1+gt_1)\gamma_0) \leqslant \mathring{\mathfrak{A}}_C((1+gt_2)\gamma_0), \qquad t_1 \leqslant t_2 \tag{51}$$

$$\mathring{\mathfrak{A}}_C((1+g\xi_1)\gamma_0) \leqslant \mathring{\mathfrak{A}}_C((1+g\eta)\gamma_0) \leqslant \mathring{\mathfrak{A}}_C((1+g\xi_2)\gamma_0) \tag{52}$$

From these inequalities and min-max principle we have

$$\mu_n[\mathring{\mathfrak{A}}_C((1+g\eta_1)\gamma_0)] \leqslant \mu_n[\mathring{\mathfrak{A}}_C((1+g\eta)\gamma_0)] \leqslant \mu_n[\mathring{\mathfrak{A}}_C((1+g\eta_2)\gamma_0)], \qquad 1 \leqslant n \leqslant N \tag{53}$$

Besides, for any $n$, $\mu_n[\mathring{\mathfrak{A}}_C((1+gt)\gamma_0)]$, $\xi_1 \leqslant t \leqslant \xi_2$, is a continuous function of $t$ in view of the inequality

$$|\mu_n[\mathring{\mathfrak{A}}_C((1+gt_1)\gamma_0)] - \mu_n[\mathring{\mathfrak{A}}_C((1+gt_2)\gamma_0)]| \leqslant \|\mathring{\mathfrak{A}}_C(g(t_2-t_1)\gamma_0)\| \tag{54}$$

which, in turn, is an elementary consequence of the min-max principle. From this continuity and the inequalities (53) we easily get

$$\mu_n[\mathring{\mathfrak{A}}_C((1+g\eta)\gamma_0)] \in \bigcup_{\eta_1 \leqslant t \leqslant \eta_2} \mu_n[\mathring{\mathfrak{A}}_C((1+gt)\gamma_0)], \qquad 1 \leqslant n \leqslant N \tag{55}$$

Now the last inclusion evidently implies the first inclusion in (48), whereas the second inclusion follows from the first one, (30), and (31). The inclusion (49) follows immediately from (31) and Lemma 3. This completes the proof of the lemma. ∎

*Proof of Theorem 1.* Let us notice first that since for any positive constant $t$: $\xi_1 \leqslant t \leqslant \xi_2$ we evidently have $\eta(x) \equiv t \in \mathscr{P}_q$, Lemma 3 and (47) imply that

$$\sigma \supseteq \sigma(\xi_1, \xi_2) \tag{56}$$

The last equation along with the relationships (49) and (47) immediately imply the equality (12). As to the equalities (13) and (14), they are easily derived from the following elementary general statement.

**Proposition 3.** Let $A$ and $B$ be bounded self-adjoint operators in a Hilbert space. Then

$$\sigma(A) \subseteq \bigcup_{-1 \leqslant t \leqslant 1} (\sigma(A) + t\,\|B\|) \tag{57}$$

This completes the proof of Theorem 1. ∎

### 3.2. Exponential Estimates for the Resolvent

In order to apply later the multiscale analysis of,[12] we need exponential estimates for the resolvent of the operator $\mathfrak{A}$. We will do this by modifying and adjusting the Combes–Thomas argument to the operator $\mathfrak{A}$. Let us consider now the relevant resolvents. Given a self-adjoint operator $A$ in either $l^2(\mathbf{Z}^d)$ or $l^2(\mathbf{Z}^3, \mathbf{C}^3)$ and $\zeta \notin \sigma(A)$, we consider the resolvents

$$G(\zeta, x, y) = (e_y, (A-\zeta)^{-1} e_x), \qquad x, y \in \mathbf{Z}^d \tag{58}$$

or

$$G(\zeta, \alpha, x, \beta, y) = (e_{\alpha, y}, (A-\zeta)^{-1} e_{\beta, x}), \qquad \alpha, \beta = 1, 2, 3, \quad x, y \in \mathbf{Z}^d \tag{59}$$

respectively. We will often drop $\alpha$ and $\beta$ in the notation of the resolvent for brevity.

**Lemma 5.** Suppose that $\eta(x)$ is a $u$-periodic function, with $u \succcurlyeq q$, which satisfies the inequalities $\xi_1 \leqslant \eta_1 \leqslant \eta(x) \leqslant \eta_2 \leqslant \xi_2$, and $A = \mathfrak{A}((1+g\eta)\gamma_0)$. Suppose also that $\operatorname{dist}\{\zeta, \sigma(\eta_1, \eta_2)\} = \delta > 0$, where $\sigma(\eta_1, \eta_2)$ is defined by (47). Then there exists a positive constant $b = b(\gamma_0, g, \xi_1, \xi_2)$ such that

$$|G(\zeta, x, y)| \leqslant 2\delta^{-1} e^{-b\delta|x-y|}, \qquad x, y \in \mathbf{Z}^d \tag{60}$$

where

$$|x| = \sum_{1 \leqslant j \leqslant d} |x_j| \tag{61}$$

Besides, the following identity is true:

$$G(\zeta, x+u, y+u) = G(\zeta, x, y), \qquad x, y \in \mathbf{Z}^d \tag{62}$$

*Proof.* For $a \in \mathbf{C}^d$ (in the case of the operator $\Lambda$, $d = 3$) let $M_a$ be the operator of multiplication by the function $M_a$, which is defined by

$$M_a(x) = e^{2\pi i a \cdot x}, \qquad x \in \mathbf{Z}^d \tag{63}$$

If we introduce an operator $A^{(a)} = M_a A M_a^{-1}$, then we obviously have

$$\Gamma^{(a)}\psi = \sum_{1 \leqslant j \leqslant d} \partial_j^{*(a)} \gamma\, \partial_j^{(a)} \psi, \qquad \Lambda^{(a)}\Psi = \nabla^{*(a)} \times (\gamma \nabla^{(a)} \times \Psi) \tag{64}$$

Clearly, the last representation implies the existence of a constant $K = K(\gamma_0, g, \xi_1, \xi_2)$ such that

$$\|\mathfrak{A} - \mathfrak{A}^{(a)}\| \leqslant K|a| \tag{65}$$

In view of (49) we have $\sigma(\mathfrak{A}) \subseteq \sigma(\eta_1, \eta_2)$, which with the conditions of the lemma implies immediately that $\|G(\zeta)\| \leqslant \delta^{-1}$. From this and the inequality (65), introducing $G(a, \zeta) = (\mathfrak{A}^{(a)} - \zeta)^{-1}$, we easily obtain

$$\|G(a, \zeta)\| \leqslant 2\delta^{-1}, \qquad |a| < \delta/(2K) \tag{66}$$

Now we notice that

$$[G(a, \zeta)](x, y) = G(\zeta, x, y) \exp\{2\pi i a \cdot (x - y)\}, \qquad x, y \in \mathbf{Z}^d \tag{67}$$

From this and the obvious inequality $|[G(a, \zeta)](x, y)| \leqslant \|G(a, \zeta)\|$ we get the inequality (60). The identity (62) follows from general statements in ref. 11 for periodic local operators. This completes the proof of the lemma. ∎

**Lemma 6.** Suppose that the condition of Lemma 5 are satisfied and that the vectors $u, v$ are such that $q \preccurlyeq u \preccurlyeq v$. Let us consider

$$\mathring{G}_{C^v}(\zeta, x, y) = [(\mathring{\mathfrak{A}}_{C^v}(\gamma) - \zeta)^{-1}](x, y), \qquad x, y \in C^v \tag{68}$$

Then the following estimate is true:

$$|\mathring{G}_{C^v}(\zeta, x, y)| \leqslant 2\delta^{-1}(1 + 2\Pi(v, \delta))\, e^{-b\delta\, |x-y|_v}, \qquad x, y \in C^v \tag{69}$$

where

$$\Pi(v, \delta) = \prod_{1 \leqslant j \leqslant d} (1 - e^{-b(\delta)\, |v_j|})^{-1}, \qquad |x - y|_v = \min_{n \in \mathbf{Z}^d} |x - y - nv| \tag{70}$$

*Proof.* The proof of the lemma is literally the same as the proof of the analogous statement (Lemma 2.15) in ref. 11. ∎

## 3.3. Wegner-Type Estimate for the Density of States

To use the methods from ref. 12 we need a Wegner-type estimate for the density of states of the operators $\Lambda$ and $\Gamma$. We obtain this estimate by a modification of Wegner's estimates (see ref. 16, Chapter II, problems 16–20).

**Lemma 7.** Let $A$, $B$, and $C$ be self-adjoint matrices of the same finite order and $E(A, d\lambda)$ be the resolution of identity, i.e., $A = \int_{\mathbf{R}} \lambda\, d\lambda$. Let $n(A, \lambda)$ be the number of eigenvalues of $A$ less than $\lambda$. Let $C$ be a nonnegative matrix of the same order. Then:

(i) For any $\lambda$

$$|n(A, \lambda) - n(B, \lambda)| \leqslant \operatorname{rank}\{A - B\} \tag{71}$$

(ii) For any continuously differentiable function $f$ with compact support

$$-\frac{\partial}{\partial t}\int n(A+tC,\mu)\,f(\mu)\,d\mu=\mathrm{Tr}\{Cf(A+tC)\} \tag{72}$$

*Proof.* The proof of (71) can be found, for instance, in ref. 16. Let us denote by $\lambda_s(t)$ and $e_s(t)$, respectively, the sets of eigenvalues and corresponding normalized eigenfunctions of self-adjoint matrices $A(t)=A+tC$. It is well known that

$$\lambda_s'=(A'e_s,e_s)=(Ce_s,e_s) \tag{73}$$

Thus, if we introduce the Heaviside function $\chi(\lambda)$, i.e., $\chi(\lambda)=1$ for $\lambda\geqslant 0$ and $\chi(\lambda)=0$ for $\lambda<0$, we can write

$$n(A+tC,\mu)=\sum_s \chi(\mu-\lambda_s) \tag{74}$$

Since the derivative of $\chi$ is the Dirac $\delta$-function, we obtain

$$-\frac{\partial}{\partial t}\int n(A+tC,\mu)\,f(\mu)\,d\mu=\int\sum_s \lambda_s'\,\delta(\mu-\lambda_s)\,f(\mu)\,d\mu=\sum_s \lambda_s' f(\lambda_s) \tag{75}$$

Using (73), we get

$$\begin{aligned}-\frac{\partial}{\partial t}\int n(A+tC,\mu)\,f(\mu)\,d\mu&=\sum_s (Ce_s,e_s)\,f(\lambda_s)\\&=\sum_s (Cf(\lambda_s)\,e_s,e_s)\\&=\sum_s (Cf(A+tC)\,e_s,e_s)\,\mathrm{Tr}\{Cf(A+tC)\}\end{aligned} \tag{76}$$

This completes the proof of the (72) and the lemma. ∎

**Lemma 8.** Let $B_s$, $s=1,\dots,m$, be nonnegative matrices of the same finite order and let $\max_{1\leqslant s\leqslant m}\operatorname{rank}\{B_s\}=r$. Let $t_s$, $s=1,\dots,m$, be nonnegative random variables such that the conditional probability distributions $p_s(dt)=\mathbf{P}\{t_s\in dt\mid t_k,\ k\neq s\}$ have densities satisfying conditions (19) and (20). Let

$$A=\sum_{1\leqslant s\leqslant m} t_s B_s \tag{77}$$

Then the measure

$$\bar{n}(A, d\lambda) = \mathbf{E}\{n(A, d\lambda)\} \tag{78}$$

is supported on the positive semiaxis and has a density such that

$$\frac{\bar{n}(A, d\lambda)}{d\lambda} \leqslant \frac{mrK}{\lambda}, \qquad \lambda \geqslant 0 \tag{79}$$

*Proof.* Let $f$ be a nonnegative, continuously differentiable function with compact support. Applying the identity (72) to the matrix $A$, we obtain

$$-\frac{\partial}{\partial t_s} \int n(A, \mu) f(\mu)\, d\mu = \mathrm{Tr}\{B_s f(A)\}, \qquad s = 1,..., m \tag{80}$$

Multiplying the last equalities by $t_s$, respectively, and summing up over $s$, we get

$$-\sum_{1 \leqslant s \leqslant m} t_s \frac{\partial}{\partial t_s} \int n(A, \mu) f(\mu)\, d\mu = \mathrm{Tr}\{A f(A)\} \tag{81}$$

Now we notice that for any $\mu$ the function $n(A, \mu)$ is a decreasing function of each argument $t_s$ since the operators $B_s$ are nonnegative. Therefore, the derivatives on the left side of the Eq. (81) are nonnegative. Taking in account this observation along with the inequalities (19), we take the expectation of both sides of the equality (81) and get

$$\begin{aligned}
&\int \lambda f(\lambda)\, \bar{n}(A, d\lambda) \\
&\quad \leqslant \sum_{1 \leqslant s \leqslant m} \int P_s(d\hat{t}_s) \int dt_s\, K(t_s)\, t_s \frac{\partial}{\partial t_s} \int -n(A, \mu) f(\mu)\, d\mu \\
&\quad = \sum_{1 \leqslant s \leqslant m} \int P_s(d\hat{t}_s) \lim_{T \to \infty} \int_0^T dt_s\, K(t_s)\, t_s \frac{\partial}{\partial t_s} \int -n(A, \mu) f(\mu)\, d\mu \\
&\quad \leqslant K \sum_{1 \leqslant s \leqslant m} \int P_s(d\hat{t}_s) \lim_{T \to \infty} \int [n(A|_{t_s=0}, \mu) - n(A|_{t_s=T}, \mu)] f(\mu)\, d\mu \\
&\quad \leqslant Kr \sum_{1 \leqslant s \leqslant m} \int P_s(d\hat{t}_s) \int f(\mu)\, d\mu = Krm \int f(\mu)\, d\mu
\end{aligned} \tag{82}$$

where $P_s(d\hat{t}_s)$ is the joint distribution of all variables $t$ except for $t_s$, since

$$0 \leqslant n(A|_{t_s=0}, \mu) - n(A|_{t_s=T}, \mu) \leqslant r \tag{83}$$

by (71). To get the last inequality in (82), we used (20). This completes the proof of the lemma. ▮

*Proof of Theorem 2.* Let $C$ be a cube in $\mathbf{Z}^d$ centered at the origin and let

$$A^C = \sum_{x \in C} t_x B_x, \qquad N^C(A, d\lambda) = (D\,|C|)^{-1}\, n(A^C, d\lambda) \tag{84}$$

Using (18), it follows from ref. 16 (see Corollary 4.3 and Theorem 4.8) that $A$ is a metrically transitive, essentially self-adjoint operator with probability 1 and

$$N(A, d\lambda) = \lim_{C \to \mathbf{Z}^d} \mathbf{E}\{N^C(A, d\lambda)\} \tag{85}$$

where the convergence is understood as the weak convergence of measures on $\mathbf{R}$. Now we notice that all operators $B_x$, being unitary equivalent, are of the same rank $r$. Now combining (85) and (79), we obtain the desired inequality (22).

To prove (23), let us notice that

$$A_{\mathscr{C}} = \sum_{x \in \mathscr{C}_x} t_x (B_x)_{\mathscr{C}} \tag{86}$$

Moreover, $(B_x)_{\mathscr{C}}$ is again a nonngative matrix with rank $\leqslant r$. Thus (23) follows from (79) by the usual argument based on Chebychev's inequality:

$$\begin{aligned} \mathbf{P}\{\operatorname{dist}(\lambda, \sigma(A_{\mathscr{C}})) \leqslant \varepsilon\} &\leqslant \mathbf{P}\left\{\int_{[\lambda-\varepsilon, \lambda+\varepsilon]} n(A_{\mathscr{C}}, d\lambda') \geqslant 1\right\} \\ &\leqslant \int_{[\lambda-\varepsilon, \lambda+\varepsilon]} \bar{n}(A_{\mathscr{C}}, d\lambda') \end{aligned} \tag{87}$$

If Assumption 1 is satisfied, then clearly the variables $\gamma_x$, $x \in \mathbf{Z}^d$, form a metrically transitive field. On other hand, if we denote by $\pi_0$ the orthogonal projection operator acting in $l^2(\mathbf{Z}^3, \mathbf{C}^3)$ [or $l^2(\mathbf{Z}^d)$] as

$$\pi_0 \Psi = \sum_{1 \leqslant \alpha \leqslant 3} (\Psi, e_{\alpha,0}) e_{\alpha,0} \qquad (\text{or } \pi_0 \psi = (\psi, e_0)\, e_0) \tag{88}$$

and define

$$B_0 \Psi = \nabla^* \times \pi_0 (\nabla \times \Psi) \qquad \left(\text{or } B_0 \psi = -\sum_{1 \leqslant j \leqslant d} \partial_j^* \pi_0\, \partial_j \psi\right) \tag{89}$$

then, if we set $t_x = \gamma_x$, it is easy to see that the operators $\Lambda$ and $\Gamma$ are of the form (17) and therefore (22) and (23) hold for these operators. This completes the proof of the theorem. ▮

## 4. PROOF OF THEOREMS 3 AND 4

Theorems 3 and 4 are proved similarly to Theorems 3 and 3′ in ref. 11. The control of the Green's functions in the singular regions is given by (23). To establish the initial probabilistic estimate for the multiscale analysis we use (69).

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